PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON THURSDAY, MARCH 21, 2024, ON Sec 9.2: DIRECTION FIELDS (SLOPE FIELDS), AUTONOMOUS D.E's, and SEC 9.3 - A SEPARABLE DE APPLICATION.

CLASS #18

Note: Some D.E's have no solutions.

Ex: The D.E. $(y')^2 = -1$

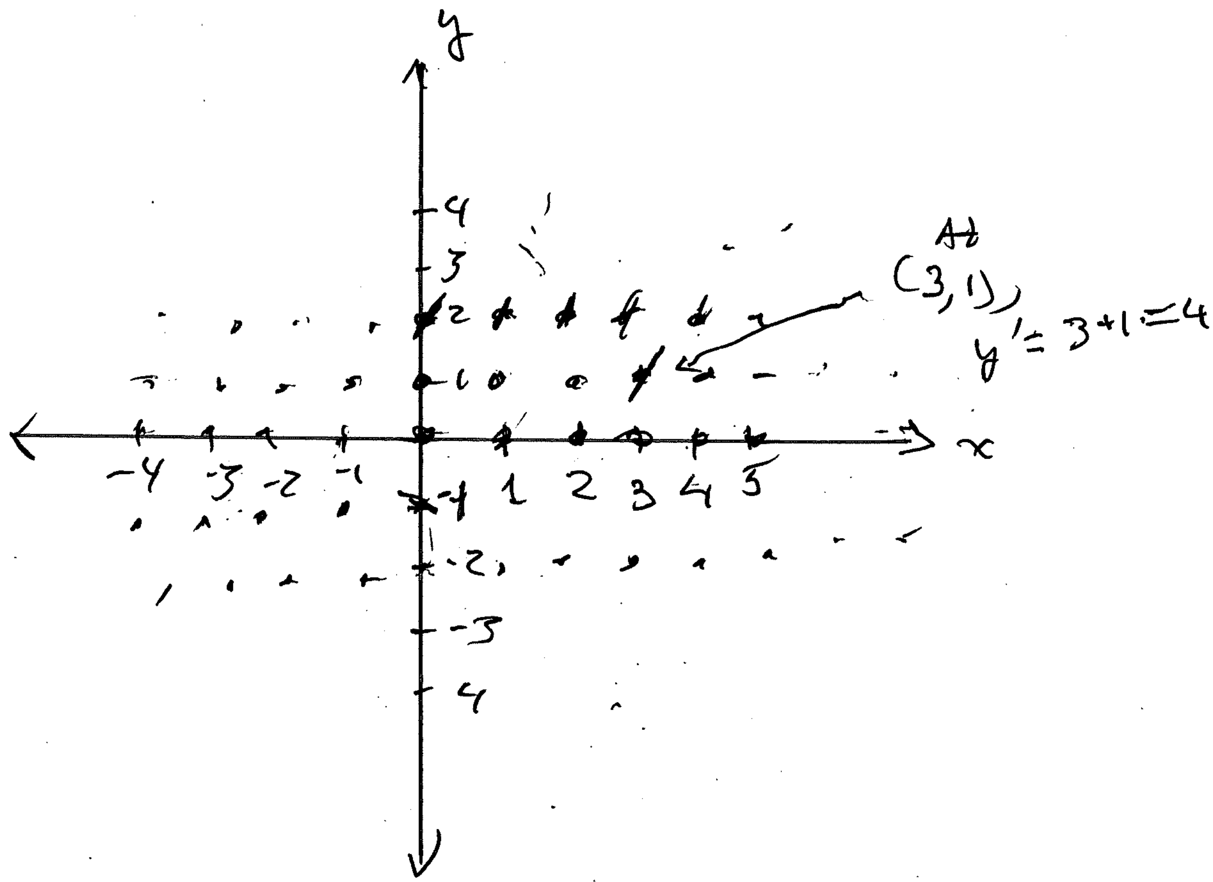
has no solution.

Even if a D.E. has a non-analytic solution, we can still learn much about the solution using Direction fields (Slope fields).

Direction Fields (Slope Fields)

Start with a given D.E,

say, $y' = x + y$.



THE DIRECTION FIELD (SLOPE FIELD) FOR THE DIFFERENTIAL EQUATION

$$y' = x + y$$

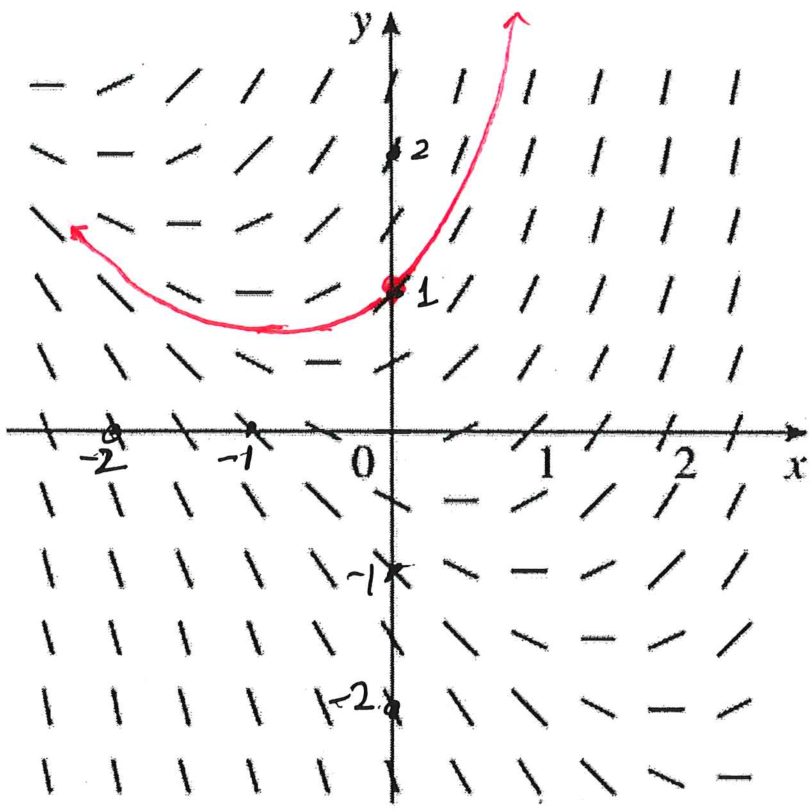


FIGURE 3
Direction field for $y' = x + y$

(p. 613)

Consider the Initial Value Problem (IVP) seeking the SOLUTION y with

$$y(0) = 1.$$

PROJECTED FIGURES AND TABLES ON DIRECTION FIELDS AND EULER'S METHOD

THE DIRECTION FIELD (p. 613)

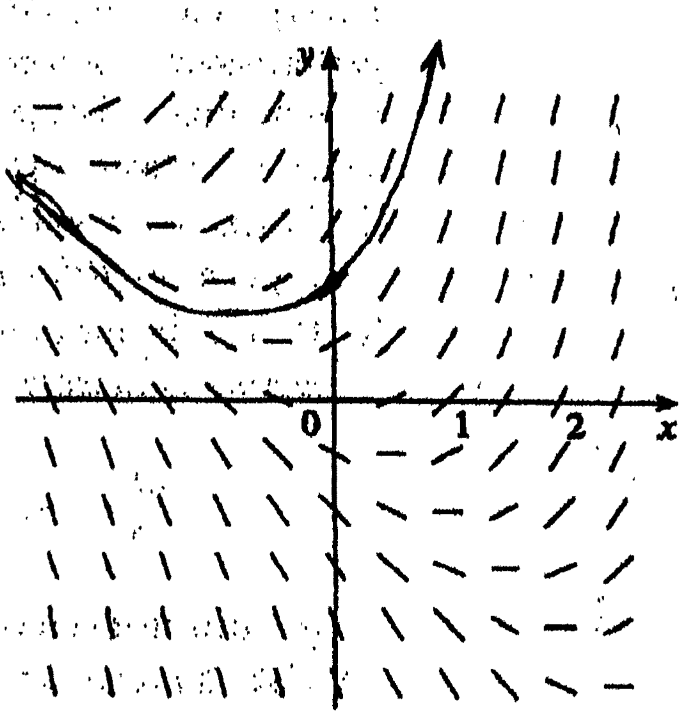


FIGURE 3
Direction field for $y' = x + y$

FOR

$y' = x + y$

At any (x_0, y_0), Slope m is $m = x_0 + y_0$

Sketch the solution curve with $y(0) = 1$

THE DIRECTION FIELD

FOR

$y' = x^2 + y^2 - 1$

(p. 614)

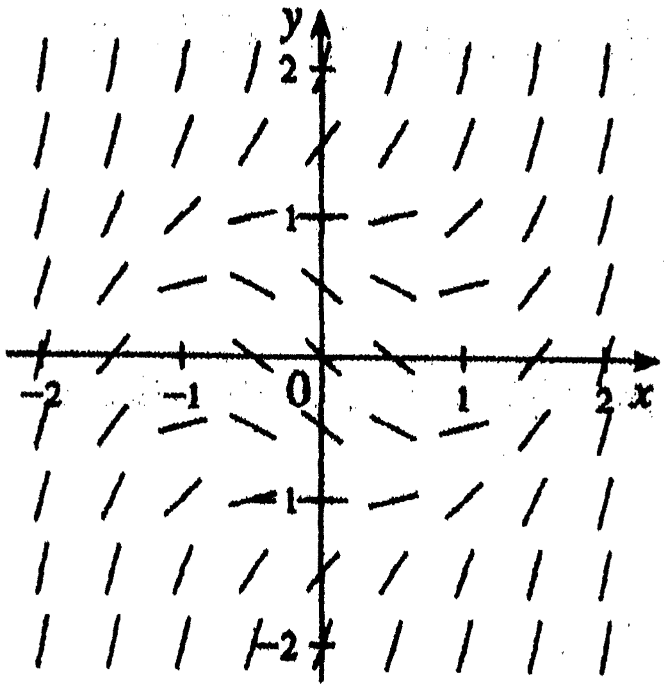


FIGURE 5

FOR

$y' = x^2 + y^2 - 1$

and

$y(0) = 0$

(p. 614)

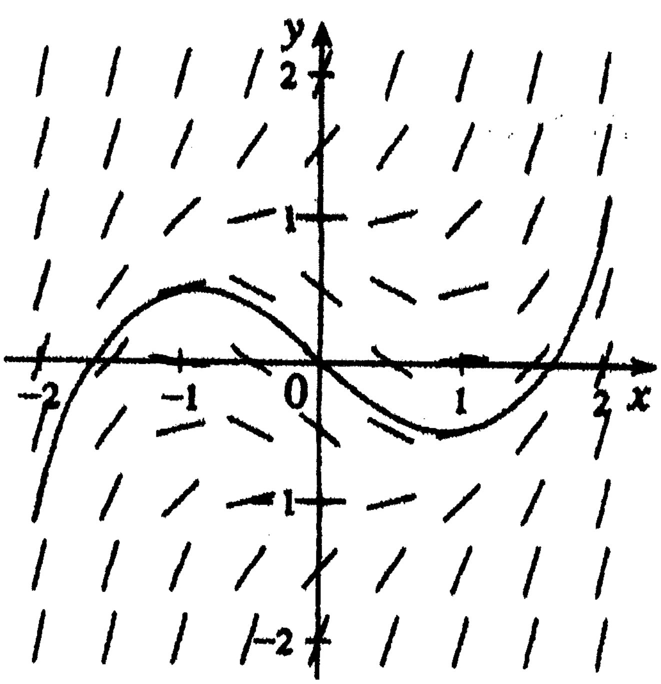


FIGURE 6

THE SLOPE FIELD
FOR
$y' = x^2 + y^2 - 1$

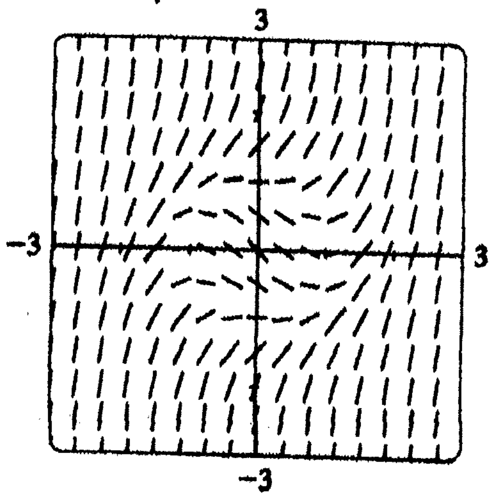


FIGURE 7
without
SOLUTION
CURVES

(p. 614)

THE SLOPE FIELD
FOR
$y' = x^2 + y^2 - 1$

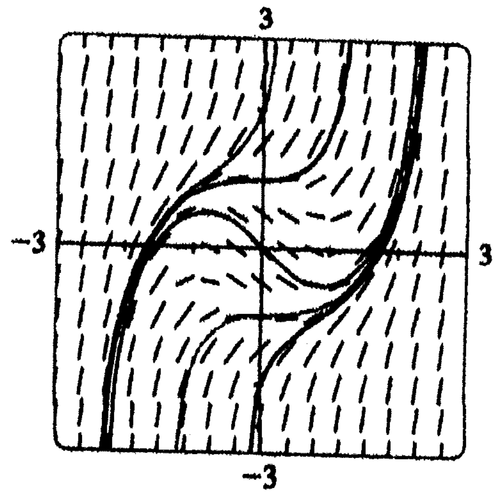


FIGURE 7

Every solution y has

$$\lim_{x \to \infty} y = \infty \text{ and}$$

$$\lim_{x \to -\infty} y = -\infty$$

The D.E. $y' = 1 - y^2$ has the form

$y' = f(y)$ where the input variable x or t does not appear in the formula for y'.

When the D.E has the form $y' = f(y)$, it is called an

AUTONOMOUS D.E.

The Solutions of a D.E, that are Constant Functions (with $y' = 0$ for all t or x and with a horizontal solution curve are called

EQUILIBRIUM SOLUTIONS

Another Autonomous D.E.

FOR
$y' = 1 - y^2$

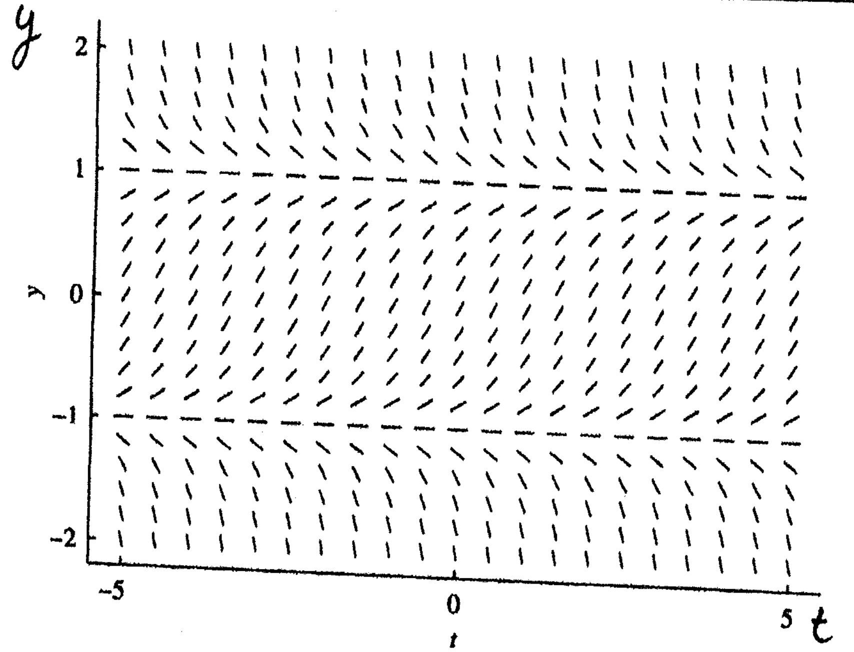
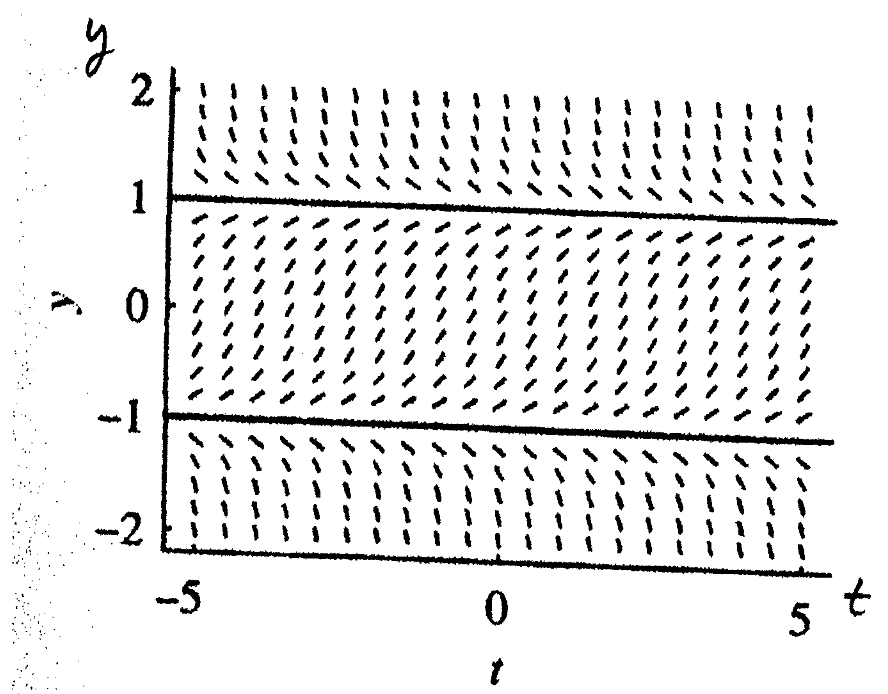


Figure 8 The direction field for the equation $y' = 1 - y^2$.

For autonomous DE $y' = f(y)$,
solve $f(y) = 0$ to find equilibrium solutions.

FOR
$y' = 1 - y^2$
$= (1+y)(1-y)$

$1 - y^2 = 0$ for
$y = -1$ and
for $y = 1$



For an autonomous D.E., $y' = f(y)$, you can locate the Equilibrium Solutions by solveing for y in the equation $f(y) = 0 = y'$, from setting the formula for y', $y' = f(y)$, equal to 0.

THE DIRECTION FIELD FOR

$$y' = 1 - y^2 = (1+y)(1-y)$$

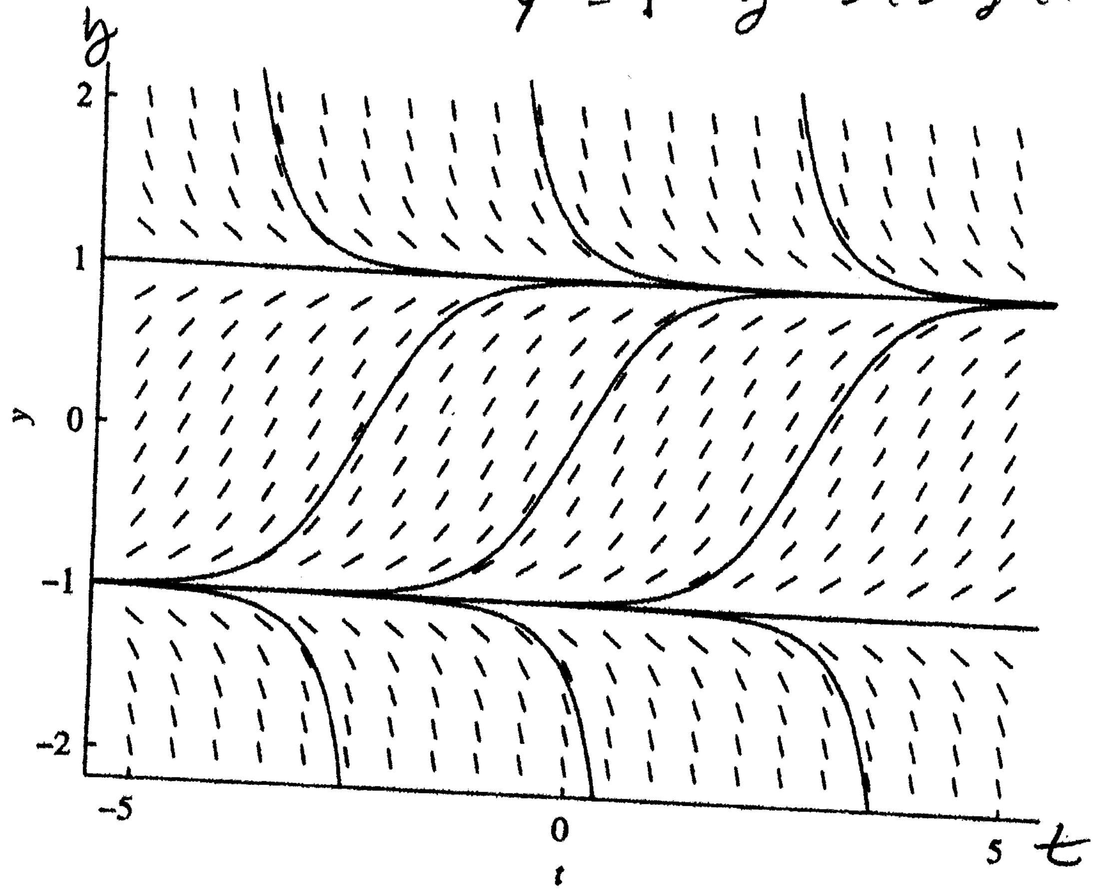


Figure 11 Typical solutions to the equation $y' = 1 - y^2$.

Solve $y' = f(y) > 0$ for $y \nearrow$

Solve $y' = f(y) < 0$ for $y \searrow$

Here: y

$y > 1$: $y' = (+)\cdot(-) < 0$, $y \searrow$

$-1 < y < 1$: $y' = (1+y)(1-y) > 0$, $y \nearrow$
$(+)\cdot(+) = (+)$

$y < -1$: $y' = (1+y)(1-y) < 0$, $y \searrow$
$(-)\,(+) = (-)$

A Separable First-Order Differential Equation Application

A full 200-gallon tank has brine (salt water) with 25 lbs of salt in the tank.

Starting at time $t = 0$ min, a brine solution with concentration 0.05 lbs / gal enters the tank at the constant rate of 10 gal/min, and the well-stirred solution is drained from the tank at the same rate, 10 gal/min.

(a) Determine the amount of salt in the tank as a function of time **t**.

(b) When does the amount of salt in the tank become 15 lbs ?

Solution: Let $y(t)$ = # of lbs of salt in the tank at time t minutes. The initial condition is $y(0) = 25$.

We will discover a D.E. in function y which has $y(t)$ as a solution.

$\frac{dy}{dt} = RATE_{IN} - RATE_{OUT}$ is in terms of $\left(\frac{\text{lbs of salt}}{\text{min}}\right)$

Brine CONCENTRATION × RATE OF Flow

$RATE_{IN} = (0.05 \text{ lbs of salt}/\text{gal})(10 \text{ gal}/\text{min}) = 0.05 \times 10$

$= 0.5$ lbs per min

CONCENTRATION × RATE OF FLOW

$RATE_{OUT} = \left(\frac{y(t) \text{ lbs}}{200 \text{ gal}}\right) \times (10 \text{ gal}/\text{min}) = \frac{10y}{200}$

$= \frac{y}{20}$ lbs of salt/min

$$\frac{dy}{dt} = 0.5 - \frac{y}{20} = \frac{10}{20} - \frac{y}{20} = \frac{1}{20}(10-y)$$

IVP: $\frac{dy}{dt} = \frac{1}{20}(10-y)$ and $y(0) = 25$

$$\frac{1}{10-y} = \frac{1}{20}dt$$

$$\int \frac{1}{10-y}dy = \int \frac{1}{20}dt = \frac{1}{20}t + C_1$$

where C_1 is any real #.

$$\int \frac{1}{10-y}dy = -\int \frac{-1}{10-y}dy = -\ln(|10-y|) + C_2$$

where C_2 is any real number.

So, $-\ln|10-y| = \frac{1}{20}t + C_3$ where $C_3 = C_2 - C_1$ and C_3 is any real #.

(this $= e^{\text{that}}$) $\ln|10-y| = -\frac{1}{20}t + C$ where $C = -C_3$; C is any real #

$$|10-y| = e^{(-\frac{1}{20}t + C)} = e^{-\frac{1}{20}t} \cdot e^{C}$$

$$|10-y| = e^{C}e^{-\frac{1}{20}t}$$, Note: $|y-10| = |10-y|$

[use the initial conditions, $y(0)=25$, to solve for e^c]

At $t=0$, $y=25$ lbs of salt, so at $t=0$

$$|10-25| = e^c \cdot e^{(-\frac{1}{20})\cdot 0} = e^c \cdot e^0 = e^c$$

$$e^c = |-15| = 15 = e^c$$

So, $|y-10| = 15\, e^{-\frac{1}{20}t}$.

So, $y-10 = \pm 15\, e^{-\frac{1}{20}t}$

$$y = 10 \pm 15 e^{-\frac{1}{20}t}$$

Since, at $t=0$, $y=25>0$

and $y = 10 \pm 15\cdot 1 = (25 \text{ or } -5)$

$y(0) = 10 \pm 15 e^0$

Since $y=25$, we select "+" and not "−"!

Solution to part (a)

$\therefore\ y = y(t) = 10 + 15 e^{-t/20}$ lbs of salt at time t.

Part (b) When does the amount of salt in the tank become 15 lbs?

Sol'n: $y(t) = 10 + e^{-t/20}$ lbs of salt.

Solve $y(t) = 15$ for t.

$$y(t) = 10 + 15e^{-t/20} = 15$$

$$15e^{-t/20} = 5$$

$$e^{-t/20} = \frac{5}{15} = \frac{1}{3}$$

$$\ln(e^{-t/20}) = \ln\left(\frac{1}{3}\right) = -\ln 3$$

$$-t/20 = -\ln 3$$

$$\frac{t}{20} = \ln 3$$

$$t = 20 \ln 3 \text{ minutes}$$

The Amount of salt in the tank at time $t = (20 \ln 3)$ minutes ≈ 21.97

is 15 lbs.